Mathematical Logics
Propositional Logic *

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Reference(s):

- **Francesco Berto, Logica da zero a Gödel, Laterza, 2018 (capitolo 1)**

**Originally by Luciano Serafini and Chiara Ghidini*
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Valid, Satisfiable, and Unsatisfiable formulas

Definition

A formula A is

Valid if for all interpretations I, I $\models A$

Satisfiable if there is an interpretation I s.t., I $\models A$

Unsatisfiable if for no interpretation I, I $\models A$

Proposition

A *Valid* $\longrightarrow A$ *satisfiable* $\longleftrightarrow A$ *not unsatisfiable*

A *unsatisfiable* $\longleftrightarrow A$ *not satisfiable* $\longrightarrow A$ *not Valid*

Valid, Satisfiable, and Unsatisfiable formulas

Proposition

if A is	*then ¬A is*
Valid	*Unsatisfiable*
Satisfiable	*not Valid*
not Valid	*Satisfiable*
Unsatisfiable	*Valid*

Chesking Validity and (un)satisfiability of a formula

Truth Table

Checking (un)satisfiability and validity of a formula A can be done by enumerating all the interpretations which are relevant for S, and for each interpretation I check if $I \models A$.

Example (of truth table)

A	B	C	$A \rightarrow (B \vee \neg C)$
true	true	true	true
true	true	false	true
true	false	true	false
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

Valid, Satisfiable, and Unsatisfiable formulas

Example

Satisfiable (Valid):

- $A \rightarrow A$
- $A \vee \neg A$
- $\neg\neg A \equiv A$
- $\neg (A \wedge \neg A)$
- $A \wedge B \rightarrow A$
- $A \rightarrow A \vee B$

Satisfiable (Non Valid):

- $A \vee B$
- $A \rightarrow B$
- $\neg (A \vee B) \rightarrow C$

Unsatisfiable (Non Valid):

- $A \wedge \neg A$
- $\neg (A \rightarrow A)$
- $A \equiv \neg A$
- $\neg (A \equiv A)$

Prove that the blue formulas are valid, that the magenta formulas are satisfiable but not valid, and that the red formulas are unsatisfiable.

Valid, Satisfiable, and Unsatisfiable sets of formulas

Definition

A set of formulas Γ is

Valid if for all interpretations I, $I \models A$ for all formulas $A \in \Gamma$

Satisfiable if there is an interpretation I, $I \models A$ for all $A \in \Gamma$

Unsatisfiable if for no interpretation I, s.t. $I \models A$ for all $A \in \Gamma$

Proposition

For any finite set of formulas Γ, (i.e., $\Gamma = \{A_1, \ldots, A_n\}$ for some $n \geq 1$), Γ is valid (resp. satisfiable and unsatisfiable) if and only if $A_1 \wedge \ldots \wedge A_n$ (resp, satisfiable and unsatisfiable).

Truth Tables: Example

Compute the truth table of $(F \vee G) \wedge \neg (F \wedge G)$.

F	G	$F \vee G$	$F \wedge G$	$\neg (F \wedge G)$	$(F \vee G) \wedge \neg (F \wedge G)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Intuitively, what does this formula represent?

Truth Tables

Recall some definitions

- Let F be a formula:
  - F is valid if every interpretation satisfies F
  - F is satisfiable if F is satisfied by some interpretation
  - F is unsatisfiable if there is no interpretation satisfying F

Truth Tables: Example (2)

Use the truth tables method to determine whether $(p \rightarrow q) \vee (p \rightarrow \neg q)$ is valid.

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The formula is valid since it is satisfied by every interpretation.

Truth Tables: Example (3)

Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with F) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	F
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

There exists an interpretation satisfying F, thus F is satisfiable.

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