

# Mathematical Logics

## Propositional Logic \*

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### Reference(s):

- Francesco Berto,  
Logica da zero a  
Gödel, Laterza, 2018  
(capitolo 1)

*\*Originally by Luciano Serafini and Chiara Ghidini  
Modified by Fausto Giunchiglia and Mattia Fumagalli*

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# Valid, Satisfiable, and Unsatisfiable formulas

## Definition

A formula  $A$  is

**Valid** if **for all interpretations**  $I, I \models A$

**Satisfiable** if **there is an interpretation**  $I$  s.t.,  $I \models A$

**Unsatisfiable** if **for no interpretation**  $I, I \models A$

## Proposition

$A \text{ Valid} \rightarrow A \text{ satisfiable} \leftrightarrow A \text{ not unsatisfiable}$

$A \text{ unsatisfiable} \leftrightarrow A \text{ not satisfiable} \rightarrow A \text{ not Valid}$

# Valid, Satisfiable, and Unsatisfiable formulas

## Proposition

<i>if A is</i>	<i>then <math>\neg A</math> is</i>
<i>Valid</i>	<i>Unsatisfiable</i>
<i>Satisfiable</i>	<i>not Valid</i>
<i>not Valid</i>	<i>Satisfiable</i>
<i>Unsatisfiable</i>	<i>Valid</i>

# Checking Validity and (un)satisfiability of a formula

## Truth Table

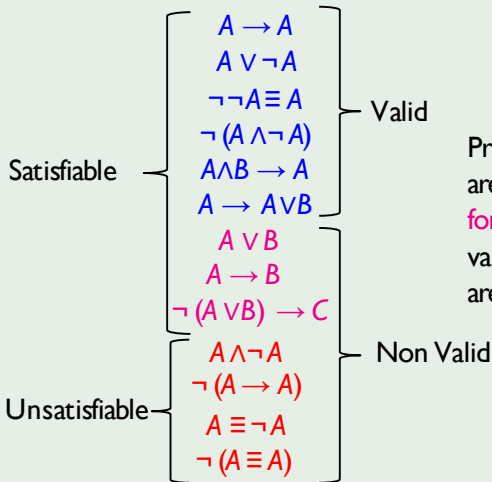
Checking (un)satisfiability and validity of a formula  $A$  can be done by enumerating all the interpretations which are relevant for  $S$ , and for each interpretation  $I$  check if  $I \models A$ .

## Example (of truth table)

$A$	$B$	$C$	$A \rightarrow (B \vee \neg C)$
true	true	true	true
true	true	false	true
true	false	true	false
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

# Valid, Satisfiable, and Unsatisfiable formulas

## Example



Prove that the blue formulas are valid, that the magenta formulas are satisfiable but not valid, and that the red formulas are unsatisfiable.

# Valid, Satisfiable, and Unsatisfiable sets of formulas

## Definition

A set of formulas  $\Gamma$  is

**Valid** if for all interpretations  $I$ ,  $I \models A$  for all formulas  $A \in \Gamma$

**Satisfiable** if there is an interpretation  $I$ ,  $I \models A$  for all  $A \in \Gamma$

**Unsatisfiable** if for no interpretation  $I$ , s.t.  $I \models A$  for all  $A \in \Gamma$

## Proposition

For any *finite set* of formulas  $\Gamma$ , (i.e.,  $\Gamma = \{A_1, \dots, A_n\}$  for some  $n \geq 1$ ),  $\Gamma$  is valid (resp. satisfiable and unsatisfiable) if and only if  $A_1 \wedge \dots \wedge A_n$  (resp. satisfiable and unsatisfiable).

# Truth Tables: Example

Compute the truth table of  $(F \vee G) \wedge \neg(F \wedge G)$ .

$F$	$G$	$F \vee G$	$F \wedge G$	$\neg(F \wedge G)$	$(F \vee G) \wedge \neg(F \wedge G)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Intuitively, what does this formula represent?



## Recall some definitions

- Let  $F$  be a formula:
  - $F$  is **valid** if every interpretation satisfies  $F$
  - $F$  is **satisfiable** if  $F$  is satisfied by some interpretation
  - $F$  is **unsatisfiable** if there is no interpretation satisfying  $F$

## Truth Tables: Example (2)

Use the truth tables method to determine whether  $(p \rightarrow q) \vee (p \rightarrow \neg q)$  is valid.

$p$	$q$	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The formula is valid since it is satisfied by every interpretation.

## Truth Tables: Example (3)

Use the truth tables method to determine whether  $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$  (denoted with  $F$ ) is satisfiable.

$p$	$q$	$r$	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	$F$
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

There exists an interpretation satisfying  $F$ , thus  $F$  is satisfiable.

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