Mathematical Logics
First Order Logic*

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5. Satisfiability, Validity, Unsatisfiability, Logical Conseguence and Logical Equivalence
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Exercises - problem

Say where these formulas are valid, satisfiable, or unsatisfiable

- $\forall x P(x)$
- $\forall x P(x) \supset \exists y P(y)$
- $\forall x . \forall y . (P(x) \supset P(y))$
- $P(x) \supset \exists y P(y)$
- $P(x) \vee \neg P(y)$
- $P(x) \wedge \neg P(y)$
- $P(x) \supset \forall x . P(x)$
- $\forall x \exists y . Q(x, y) \supset \exists y \forall x Q(x,y)$
- $x = x$
- $\forall x . P(x) \equiv \forall y . P(y)$
- $x = y \supset \forall x . P(x) \equiv \forall y . P(y)$
- $x = y \supset (P(x) \equiv P(y))$
- $P(x) \equiv P(y) \supset x = y$

Exercizes - Solution

$\forall x P(x)$	Satisfiable
$\forall x P(x) \supset \exists y P(y)$	Valid
$\forall x . \forall y . (P(x) \supset P(y))$	Satisfiable
$P(x) \supset \exists y P(y)$	Valid
$P(x) \vee \neg P(y)$	Satisfiable
$P(x) \wedge \neg P(y)$	Satisfiable
$P(x) \supset \forall x . P(x)$	Satisfiable
$\forall x\ \exists y . Q(x, y) \supset \exists y\ \forall x Q(x, y)$	Satisfiable
$x = x$	Valid
$\forall x . P(x) \equiv \forall y . P(y)$	Valid
$x = y \supset \forall x . P(x) \equiv \forall y . P(y)$	Valid
$x = y \supset (P(x) \equiv P(y))$	Valid
$P(x) \equiv P(y) \supset x = y$	Satisfiable

Exercizes - Expressing properties in FOL

What is the meaning of the following FOL formulas?

1. $\exists x\ (bought(Frank, x) \wedge dvd\ (x))$
2. $\exists x.bought(Frank, x)$
3. $\forall x.(bought(Frank, x) \rightarrow bought(Susan, x))$
4. $(\forall x.bought(Frank, x)) \rightarrow (\forall x.bought(Susan, x))$
5. $\forall x\ \exists y.bought(x, y)$
6. $\exists x\ \forall y.bought(x, y)$

1. ”Frank bought a dvd.”
2. ”Frank bought something.”
3. ”Susan bought everything that Frank bought.”
4. ”If Frank bought everything, so did Susan.”
5. ”Everyone bought something.”
6. ”Someone bought everything.”

Exercizes - expressing properties in FOL

For each property write a formula expressing the property, and for each formula write the property it formalises.

- Every Man is Mortal

  $\forall x.Man(x) \supset Mortal\ (x)$
- Every Dog has a Tail

  $\forall x.Dog\ (x) \supset \exists y\ (PartOf\ (x, y) \wedge Tail\ (y))$
- There are two dogs

  $\exists x, y\ (Dog\ (x) \wedge Dog\ (y\ ) \wedge x \neq y\ )$
- Not every dog is white

  $\neg\forall x.Dog\ (x) \supset White(x)$
- $\exists x.Dog\ (x) \wedge \exists y.Dog\ (y)$

  There is a dog
- $\forall x, y\ (Dog\ (x) \wedge Dog\ (y) \supset x = y\ )$

  There is at most one dog

Exercizes – problem

Define an appropriate language and formalize the following sentences using FOL formulas.

1. All Students are smart.
2. There exists a student.
3. There exists a smart student.
4. Every student loves some student.
5. Every student loves some other student.
6. There is a student who is loved by every other student.
7. Bill is a student.
8. Bill takes either Analysis or Geometry (but not both).
9. Bill takes Analysis and Geometry.
10. Bill doesn’t take Analysis.
11. No students love Bill.

Exercizes - solution

1. $\forall x.(Student(x) \longrightarrow Smart(x))$
2. $\exists x.Student(x)$
3. $\exists x.(Student(x) \land Smart(x))$
4. $\forall x.(Student(x) \longrightarrow \exists y.(Student(y) \land Loves\ (x, y))$
5. $\forall x.(Student(x) \longrightarrow \exists y.(Student(y) \land \neg(x = y) \land Loves\ (x, y)))$
6. $\exists x.(Student(x) \land \forall y.(Student(y) \land \neg(x = y) \longrightarrow Loves\ (y , x)))$
7. $Student(Bill)$
8. $Takes\ (Bill, Analysis)\ \boxed{\leftrightarrow}\ \neg Takes\ (Bill, Geometry)$
9. $Takes\ (Bill, Analysis) \land Takes\ (Bill, Geometry)$
10. $\neg Takes\ (Bill, Analysis)$
11. $\neg \exists x.(Student(x) \land Loves\ (x, Bill))$

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