Mathematical Logics
Modal Logic: K and more*

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2. Modal K (Hilbert calculus)
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R is reflexive

The axiom T

If a frame is reflexive (we say that a frame has a property, when the relation R has such a property) then the formulas

$$\textbf{T} \quad \Box\varphi \supset \varphi$$

holds. (Or alternatively $\varphi \supset \Diamond\varphi$.)

A FRAME F IS REFLEXIVE $\Rightarrow F \models \Box\varphi \supset \varphi$

$$\forall w.\ R(w,w)$$

NOTE

$$\Box\varphi \supset \varphi \ \text{iff}\ \neg\Diamond\neg\varphi \supset \varphi \ \text{iff}$$

$$\Diamond\neg\varphi \vee \varphi$$

take $\varphi = \neg\Psi$

$$\Diamond\Psi \vee \neg\Psi \ \text{iff}\ \Psi \supset \Diamond\Psi$$

NOTE: THE LOGIC OF KNOWLEDGE

R is reflexive - soundness

Let M be a model on a reflexive frame $F = (W, R)$ and w any world in W. We prove that $M, w \models \Box\varphi \supset \varphi$.

1. Since R is reflexive then wRw
2. Suppose that $M, w \models \Box\varphi$ (Hypothesis)
3. From the satisfiability condition of $\Box$, $M, w \models \Box\varphi$, and wRw imply that
4. $M, w \models \varphi$ (Thesis)

   Since from (Hypothesis) we have derived (Thesis), we can conclude that

   $M, w \models \Box\varphi \supset \varphi$.

R is reflexive - completeness

Suppose that a frame $F = (W, R)$ is not reflexive.

1. If R is not reflexive then there is a $w \in W$ which does not access to itself. I.e., for some $w \in W$ it does not hold that wRw.
2. Let M be any model on F, and let φ be the propositional formula p. Let V the set p true in all the worlds of W but w where p is set to be false.
3. From the fact that w does not access to itself, we have that in all the worlds w accessible from w, p is true, i.e, $\forall w', wRw', M, w' \vDash p$.
4. Form the satisfiability condition of $\Box$ we have that M,

   $w \vDash \Box p$.
5. since $M, w \nvDash p$, we have that $M, w \nvDash \Box p \supset p$.

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