Mathematical Logics
Modal Logic: K and more*

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R is euclidean and reflexive

The axiom 5

If a frame is euclidean and reflexive then the formula

$$\mathbf{5} \quad \Diamond\varphi \supset \Box\Diamond\varphi$$

holds.

NOTE = Euclidean and reflexive iff equivalence relation

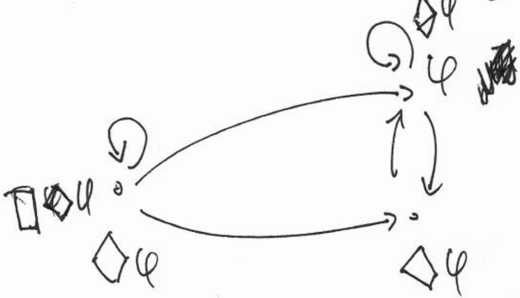
An equivalence relation is a binary relation that is reflexive, symmetric and transitive. That is, for any objects a, b, and c:

- R(a , a) (reflexive property),
- if R(a , b) then R(b , a) (symmetric property),
- if R(a , b) and R(b , c) then R(a , c) (transitive property).

IF A FRAME F IS EUCLIDIAN AND REFLEXIVE THEN

$$F \models \Diamond\varphi \supset \Box\Diamond\varphi$$

$$\underbrace{\forall w . R(w,w)}_{T} \; ; \; \underbrace{\forall w,v,u. \left(R(w,v) \land R(w,u) \supset R(v,u) \right)}_{5}$$



$$\Diamond\varphi \supset \Box\Diamond\varphi$$

R is euclidean and reflexive - soundness

Let M be a model on a euclidean frame $F = (W, R)$ and w any world in W. We prove that $M, w \models \Diamond\varphi \supset \Box\Diamond\varphi$.

1. Suppose that $M, w \models \Diamond\varphi$ (Hypothesis).
2. The satisfiability condition of $\Diamond$ implies that there is a world w^l accessible from w such that $M, w^l \models \varphi$.
3. We have to prove that $M, w \models \Box\Diamond\varphi$ (Thesis)
4. From the satisfiability condition of $\Box$, this is equivalent to prove that for all world w^{ll} accessible from w $M, w^{ll} \models \Diamond\varphi$,
5. let w^{ll} be any world accessible from w. The fact that R is euclidean, the fact that wRw^l implies that $w^{ll}Rw^l$.
6. Since $M, w^l \models \varphi$, the satisfiability condition of $\Diamond$ implies that $M, w^{ll} \models \Diamond\varphi$.
7. and therefore $M, w \models \Box\Diamond\varphi$. (Thesis)
8. Since from (Hypothesis) we have derived (Thesis), we can conclude that $M, w \models \Box\varphi \supset \Box\Diamond\varphi$.

R is euclidean and reflexive - completeness

Suppose that a frame F = (W, R) is not euclidean.

1. If R is not euclidean then there are three worlds $w, w', w'' \in W$, such that wRw', wRw'' but not $w'Rw''$.
2. Let M be any model on F, and let φ be the propositional formula p. Let V the set p false in all the worlds of W but w' where p is set to be true.
3. From the fact that w'' does not access to w', and in all the other worlds p is false, we have that $w'' \not\models \Diamond p$
4. this implies that $M, w \not\models \Box\Diamond p$.
5. On the other hand, we have that wRw', and $w' \models p$, and therefore $M, w \models \Diamond p$. $M, w \not\models \Box p \supset \Box\Box p$.
6. In summary: $M, w \not\models \Box\Diamond p$, and $M, w \models \Diamond P$; from which we have that M, $w \not\models \Diamond p \supset \Box\Diamond p$.

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