Mathematical Logics
Description Logic: Tbox and Abox

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Tbox – Reasoning services

Given two class-propositions P and Q, we have the following reasoning problems

- Satisfiability w.r.t. T $T \models P$?
- Subsumption $T \models P \sqsubseteq Q$? $T \models Q \sqsubseteq P$?
- Equivalence $T \models P \sqsubseteq Q$ and $T \models Q \sqsubseteq P$?
- Disjointness $T \models P \sqcap Q \sqsubseteq \bot$?

Satisfiability with respect to a TBox T

A concept P is satisfiable w.r.t. a terminology T, if there exists (or for all) an interpretation I with $I \models \theta$ for all $\theta \in T$, and such that $I \models P$, namely I(P) is not empty

In this case we also say that I is a model for P

In other words, the interpretation I not only satisfies P, but also complies with all the constraints in T

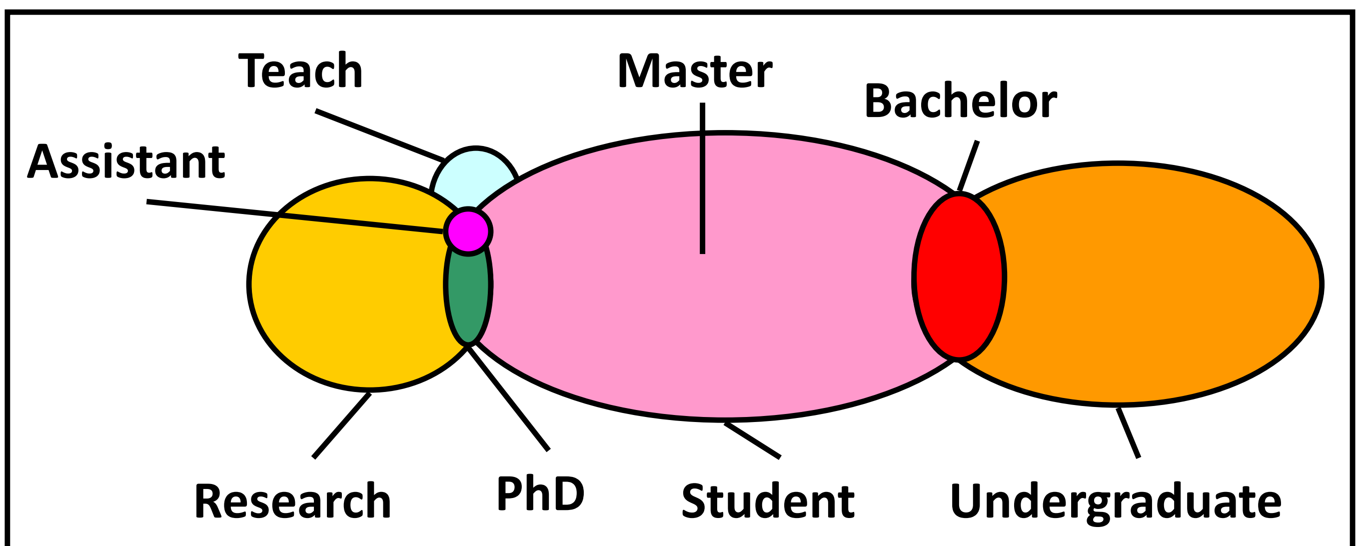
Satisfiability with respect to a TBox T

Suppose we describe the students in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

The TBox is satisfiable. A possible model is:



In this model the two concepts **Bachelor** and **Assistant** are satisfiable w.r.t. T, while the concept **Assistant ⊓ Bachelor** is not.

TBox reasoning: subsumption

Let T be a TBox. A Subsumption problem (with respect to T): is defined as follows

$$T \models P \sqsubseteq Q \; (P \sqsubseteq_T Q)$$

A concept P is subsumed by a concept Q with respect to T if $I(P) \subseteq I(Q)$ for every model I of T

NOTE: Subsumption reduces to entailment and validity (when T empty)

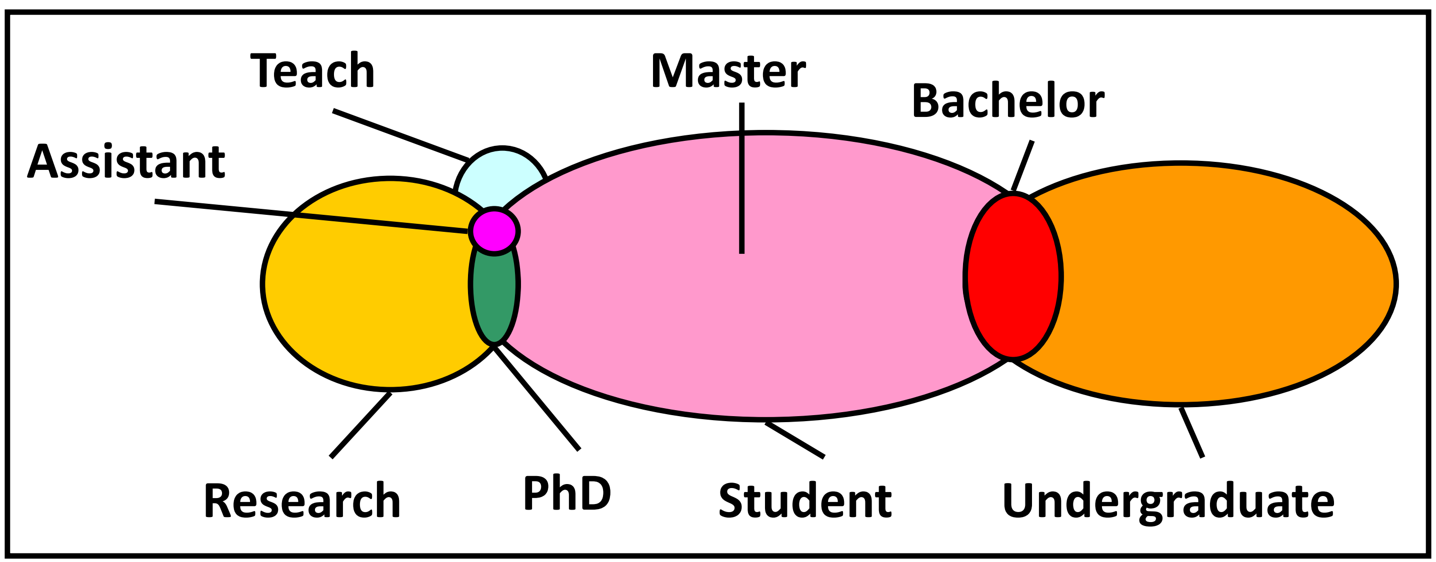
Subsumption with respect to a TBox T (1)

Suppose we describe the students in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

T $\models$ PhD $\sqsubseteq$ Student



Subsumption with respect to a TBox T (2)

PhD $\sqsubseteq$ Student

Proof:

PhD

$\equiv$ Master $\sqcap$ Research

$\equiv$ (Student $\sqcap$ $\neg$ Undergraduate) $\sqcap$ Research

$\sqsubseteq$ Student

TBox reasoning: equivalence

Let T be a TBox. An Equivalence problem (with respect to T) is defined as follows:

$$(T \vDash P \equiv Q) \ (P \equiv_T Q)$$

Two concepts P and Q are equivalent with respect to T if I(P) = I(Q) for every model I of T.

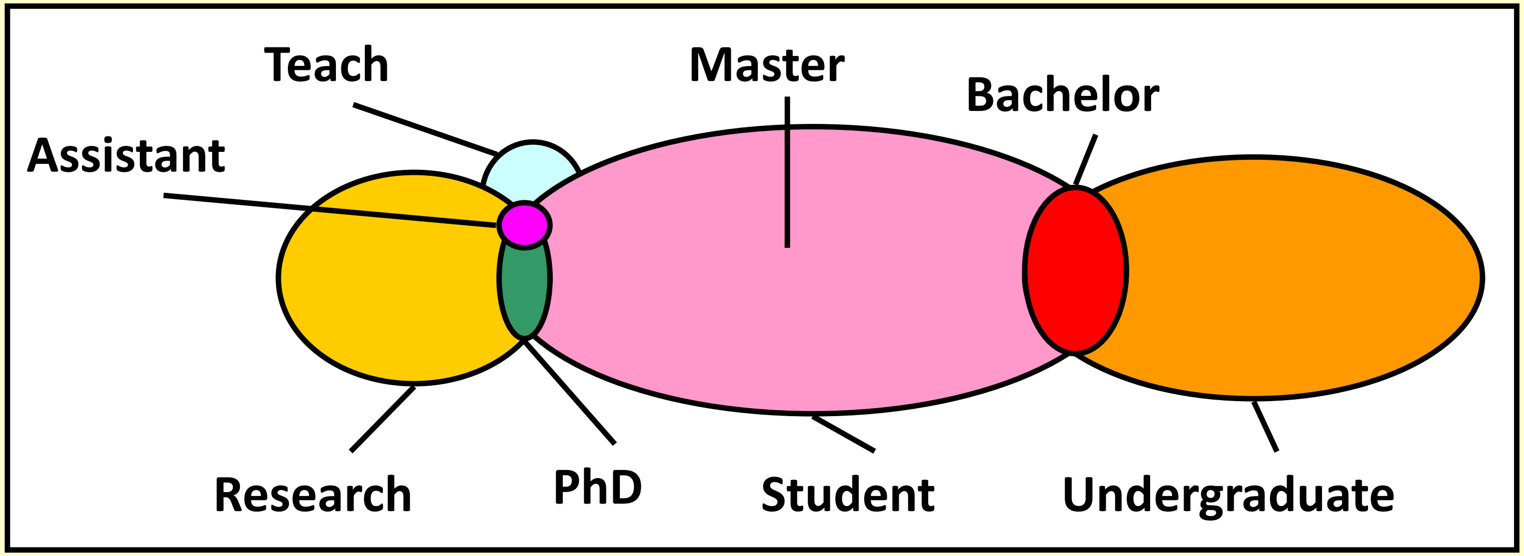
Equivalence with respect to a TBox T (I)

Suppose we describe the students in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

T $\models$ Student $\equiv$ Bachelor $\sqcup$ Master



Equivalence with respect to a TBox T (2)

Student $\equiv$ Bachelor $\sqcup$ Master

<u>Proof</u>:

Bachelor $\sqcup$ Master

$\equiv$ (Student $\sqcap$ Undergraduate) $\sqcup$ Master

$\equiv$ (Student $\sqcap$ Undergraduate) $\sqcup$ (Student $\sqcap$ $\neg$ Undergraduate)

$\equiv$ Student $\sqcap$ (Undergraduate $\sqcup$ $\neg$Undergraduate)

$\equiv$ Student $\sqcap \top$

$\equiv$ Student

Tbox reasoning: disjointness

Let T be a TBox. A Disjointness problem (with respect to T) is defined as follows:

$$T \models P \sqcap Q \sqsubseteq \bot \ (P \sqcap Q \sqsubseteq_T \bot)$$

Two concepts P and Q are disjoint with respect to T if their intersection is empty, $I(P) \cap I(Q) = \emptyset$, for every model I of T.

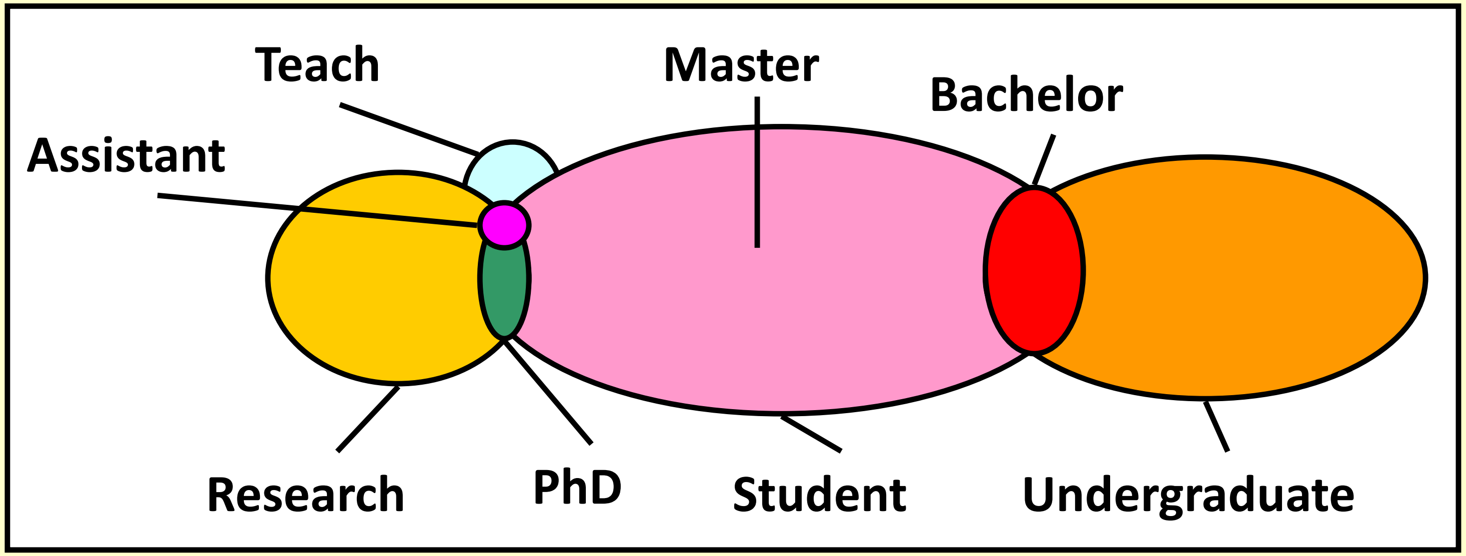
Disjointness with respect to a TBox T (1)

Suppose we describe the students in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

T $\models$ Undergraduate $\sqcap$ Assistant $\sqsubseteq \bot$



Disjointness with respect to a TBox T (2)

It can be proved showing that:

$T \models \text{Undergraduate} \sqcap \text{Assistant} \sqsubseteq \bot$

<u>Proof</u>:

$$\begin{aligned}
\text{Undergraduate} \sqcap \text{Assistant} & \\
& \sqsubseteq \neg\text{Teach} \sqcap \text{Assistant} \\
& \equiv \neg\text{Teach} \sqcap \text{PhD} \sqcap \text{Teach} \\
& \equiv \bot \sqcap \text{PhD} \\
& \equiv \bot
\end{aligned}$$

Exercise

Suppose we describe the students in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

Is Bachelor $\sqcap$ PhD satisfiable?

NO!

Consider the following propositions:

Assistant, Student, Bachelor, Teach, PhD, Master $\sqcap$ Teach

- Which pairs are subsumed/supersumed?
- Which pairs are disjoint?

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